**General Consequences of the Postulates**

Now let’s examine some of the general consequences of these postulates before we start analyzing any specific systems.

**Statistics of an observable**

According to the second set of postulates, if we measure the A of a particle, the probability that we get a particular eigenvalue, *a*, is given by:



where |a> is the eigenvector of the operator Â with eigenvalue *a*. So this is the probability distribution of the eigenvalue *a*. Let’s examine some features of the probability distribution. For instance, what is the average value of Athat we can expect upon measurement. This the expectation of *a*, which is defined in probability and statistics class as:



Recognizing the ( ) as the identity operator, Î, we can say that the expectation of A is:



Now let’s ask, what is the standard deviation of A? The standard deviation is defined as:



You might recognize this as the square root of the variance. Anyway, we can put this in nicer form.



So we have the relatively benign expression:



**Example: Probability distribution of position**

Suppose we have a particle with the wavefunction:



What is the probability distribution of its position (in 1D). What is the most likely value to result upon measurement? What is the average value. What is the standard deviation?

Well, the probability distribution of the position is (in 1D)



The most probable value of x is the one which maximizes P(x). Without taking a derivative, this is clearly x = 0. The expectation of x is:



This formula should make sense because it is just ∫dx P(x)x. So inserting ψ(x) we get:



The standard deviation of the measurements of x (on similar wavefunctions) is:



The first term we need to evaluate is . Doing the same thing as we did for  to put it in integral form, we get:



So then the standard deviation is:



**Heisenberg uncertainty principle**

If the operators corresponding to two observables commute, then you’ll recall that we can simultaneously diagonalize both operators, meaning that we can find a set of vectors which are eigenvectors for both operators. That being the case, a particle can be in a state with a specific value of A and B, say |ψab> so that when we measure the A and B we will get unambiguously the eigenvalues a and b.

However if two observables do not commute, then we cannot simultaneously diagonalize them and it is impossible that the particle exists in a state in which both are known with certainty. Therefore, for whatever state the particle is in, there will necessarily be some uncertainty in A and B, i.e.,, some standard deviation ΔA and ΔB of the probability distribution P(a) and P(b) of the respective eigenvalues. Heisenberg’s uncertainty principle put a lower bound on the product of the uncertainties ΔA and ΔB. It states:



**Proof**

The proof follows from the Schwartz inequality, namely that:



We’ll define our vector as:



Then the inequality states that:



Now we recognize that the expectation of the commutator is imaginary since:



So if any number is equal to the negative of its complex conjugate, it must be imaginary since



the only way to make the equality work is if x = 0. Working out the same process, we can show that



and so it must be real. Calculating the modulus of this complex number,



we can say that:



which has proved the inequality. So we cannot know two non-commuting observables with simultaneous absolute precision. However if they commute, then it is possible.

**Example: position-momentum uncertainty relation**

The position operator and momentum operator do not commute as is demonstrated below:



So we can say that:



where Î is the identity operator of course. Usually, since Î just acts like the number 1, it is left off the equation. So this is a rather famous equation in its own right, and in graduate quantum mechanics courses is used as to justify the expression for the momentum operator we’ve previously ‘derived’. For our purposes though, we use this relationship to recognize that we cannot know the position and momentum of a particle simultaneously, since the commuter isn’t 0. How much can we minimize the uncertainty? According to the Heisenberg uncertainty principle, the minimum uncertainty in both is:



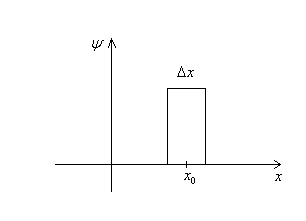
so we have:



Remember that from physical arguments in lecture 2 we found that ΔxΔpx ≈ h, which is very close to this expression. Our previous argument resulted in a different formula because we did not rigorously define Δx or Δpx. Here they are rigorously defined as the standard deviation of the probability distribution of x and px respectively.

**Example: position-momentum uncertainty relation**

Suppose we have an a position wavefunction ψ(x) with position uncertainty Δx. Say something like a box of width Δx centered about the x­0.



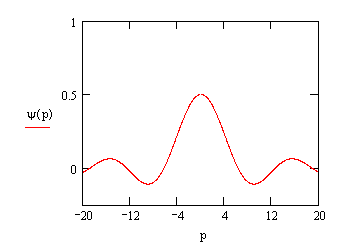
What is its momentum uncertainty? For this we must calculate ψ(p). Well,



So we have,



which looks something like (for Δx = 1)



and we see that this is a decaying (oscillating) function of p, with a width determined rougly by where the function hits zero first on either side of the pick. This is where the argument is -π and π respectively. This happens when p ~ -2π/Δx, and p ~ 2π/Δx respectively. So,



which when combined gives,



which is consistent with the Heisenberg uncertainty relation of course. On a more extreme note, lets observe that if we are in an eigenstate of **p**, then |ψ|2 = 1, in which case there is a uniform probability of being anywhere. So the uncertainty in position is ∞, which again preservers the inequality.

**Feynman-Hellman Theorem**

Last, let’s quickly note an identity that is quite useful in many contexts, it is the Feynman-Hellman theorem. Suppose we have an operator, A, which depends on some parameter, λ, and has exact eigenstates ψn(λ), and eigenvalues an(λ). Then we can certainly say:



And it follows that:



where in the third term we use the Hermiticity of the operator . And then since the eigenvalues must be real, we can say,



d(1)/dλ is zero of course. And so we have:



Another useful result is, which can be proved analogously, is:



for m ≠ n.